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Turbulent dynamo action at low magnetic Reynolds number

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The effect of turbulence on a magnetic field whose length-scale L is initially large compared with the scale l of the turbulence is considered. There are no external sources for the field, and in the absence of turbulence it decays by ohmic dissipation. It is assumed that the magnetic Reynolds number $R_m = u_0 l/\lambda$ (where u_0 is the root-mean-square velocity and λ the magnetic diffusivity) is small. It is shown that to lowest order in the small quantities l/L and R_m, isotropic turbulence has no effect on the large-scale field; but that turbulence that lacks reflexional symmetry is capable of amplifying Fourier components of the field on length scales of order $R_m^{-2} l$ and greater. In the case of turbulence whose statistical properties are invariant under rotation of the axes of reference, but not under reflexions in a point, it is shown that the magnetic energy density of a magnetic field which is initially a homogeneous random function of position with a particularly simple spectrum ultimately increases as $t^{-\frac{1}{2}}\exp(\alpha^2 t/2\lambda^3)$ where $\alpha(=O(u_0^2 l))$ is a certain linear functional of the spectrum tensor of the turbulence. An analogous result is obtained for an initially localized field.

1. Introduction

A theory that is likely to be of the greatest significance in geomagnetism and in cosmical electrodynamics has been developed recently by Steenbeck, Krause & Rädler (1966), Steenbeck & Krause (1966, 1967), Rädler (1968) and Krause (1968). The theory is concerned with the effect of a turbulent velocity field on a magnetic field distribution in an electrically conducting fluid, it being supposed that there is no external source of magnetic field, the only source being the electric current distribution within the fluid itself.

A principal conclusion of these authors is that dynamo action (i.e. systematic transfer of energy from the velocity field to the magnetic field) will occur provided only that the statistical properties of the turbulence lack reflexional symmetry, i.e. are not invariant under a change from a right-handed to a left-handed frame of reference. The arguments can be justified with some degree of rigour only when the magnetic Reynolds number $R_m = \mu\sigma l u_0$ satisfies the condition

$$R_m \ll 1; \tag{1.1}$$

here, μ is the magnetic permeability of the fluid, σ its electrical conductivity, l the length-scale characteristic of the energy-containing eddies and u_0 the r.m.s.

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turbulent velocity. It seems quite likely however that a lack of reflexional symmetry is a sufficient condition for dynamo action irrespective of the order of magnitude of R_m.

The above result is in striking contrast with previous predictions concerning turbulent dynamo action. For example, Syrovatsky (1959) argued (on the basis of ideas put forward by Biermann & Schlüter 1950) that dynamo action will occur only if R_m is greater than some number of order unity; Batchelor (1950) argued on the basis of the 'vorticity analogy' that sustained dynamo action will occur only if the magnetic diffusivity $\lambda = (\mu\sigma)^{-1}$ is smaller than the kinematic viscosity of the fluid, i.e. only if $R_m > R$ where $R(\gg 1)$ is the turbulent Reynolds number; and Saffman (1963) argued that dynamo action might not occur under *any* circumstances due to an accelerating ohmic diffusion effect which occurs when the length scale of a convected magnetic field is systematically decreased. These authors concentrated on the tendency for magnetic energy to be swept towards higher wave-numbers, in spectral terminology, but they underestimated the potential importance of the possible 'leak back' to lower wave-numbers which can occur and which turns out to be at the heart of the Steenbeck, Krause & Rädler mechanism. The possibility of such a leak back was recognized but not quantitatively analyzed by Kraichnan & Nagarajan (1967). Some of the relevant arguments were reviewed by Moffatt (1961).

A first version of the present paper was submitted for publication before I was aware of the existence of the above series of papers by Steenbeck *et al.* There is a considerable overlap between the work described in §§2 and 3 and the work of these authors. The notation and the detailed method of analysis are different, but the conclusions are substantially the same. The treatment given here is certainly more compact than that given by Steenbeck *et al.*, and since the results have been by no means widely recognized or accepted, it is felt that this complementary approach is well justified. The physical interpretation of the 'helicity effect' given in §4 has not been given previously, and the demonstration of dynamo action given in §§5 and 6 is definitely simpler and more convincing than the arguments (based on more complicated models) given by Steenbeck *et al.*

2. The averaged effect of the turbulence on the magnetic field

We consider a fluid of infinite extent in a state of homogeneous turbulent motion with zero mean velocity. It will be further supposed that the mean properties of the turbulence (e.g. correlation tensors) do not change with time, i.e. the turbulence is stationary as well as homogeneous; departures from these idealized conditions can be incorporated at a later stage. The theory that follows is essentially a kinematical one, in which all statistical properties of the turbulence (determined by *dynamical* processes) are assumed known, and the evolution of a passive vector field, which is both convected and diffused, is investigated.

An electric current distribution $\mathbf{J}(\mathbf{x}, t)$ in the fluid will give rise to a magnetic field distribution $\mathbf{B}(\mathbf{x}, t)$ satisfying

$$\nabla \wedge \mathbf{B} = \mu \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0, \tag{2.1}$$

and this develops according to the equation

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) + \lambda \nabla^2 \mathbf{B}, \tag{2.2}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the turbulent velocity field. The condition $R_m \ll 1$ ensures that magnetic fluctuations on scales of order l and less will tend to be rapidly suppressed. Suppose however that, at some initial instant $t = 0$, we have a magnetic field $\mathbf{B}(\mathbf{x}, 0)$ whose characteristic length scale L satisfies

$$L \gg l. \tag{2.3}$$

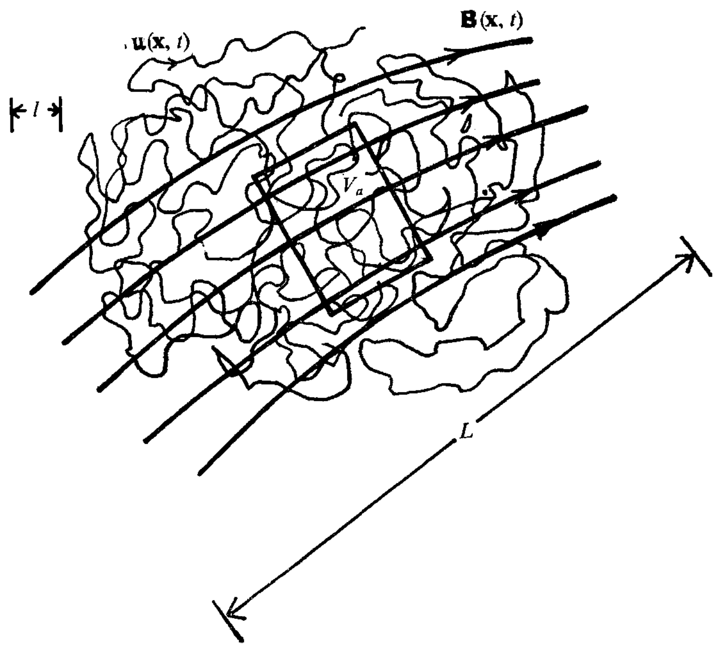

FIGURE 1. Schematic picture of the velocity field $\mathbf{u}(\mathbf{x}, t)$ on the scale l and the initial magnetic field $\mathbf{B}(\mathbf{x}, 0)$ on the scale L. The box V_a is large enough to contain a large number of turbulent eddies, but small enough for the field $\mathbf{B}(\mathbf{x}, 0)$ to be approximately uniform inside it.

Figure 1 shows the sort of picture that is envisaged. The light line represents an instantaneous streamline of the field $\mathbf{u}(\mathbf{x}, t)$. The heavy lines represent the lines of force of $\mathbf{B}(\mathbf{x}, 0)$, with mean curvature $O(L^{-1})$. The field $\mathbf{B}(\mathbf{x}, 0)$ may be a localized field decreasing to zero outside some finite region, or it may itself be a random function of position, possibly homogeneous. In the absence of any turbulence, such a field would decay according to the diffusion equation

$$\partial \mathbf{B}/\partial t = \lambda \nabla^2 \mathbf{B}, \tag{2.4}$$

in a time of order t_d where

$$t_d = L^2/\lambda. \tag{2.5}$$

In the presence of the turbulence, the governing equation becomes (2.2), the term $\nabla \wedge (\mathbf{u} \wedge \mathbf{B})$ representing the inductive effect of flow across the magnetic field. This term will undoubtedly generate magnetic fluctuations on the scale l

(and smaller). It is then appropriate to treat the total field $\mathbf{B}(\mathbf{x}, t)$ as the sum of a field $\mathbf{B}_0(\mathbf{x}, t)$ on the scale L and the generated field $\mathbf{b}(\mathbf{x}, t)$ on scales of order l:

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t). \tag{2.6}$$

If equation (2.2) is averaged throughout a box V_a of side a satisfying

$$l \ll a \ll L, \tag{2.7}$$

and the average is denoted by an overbar, then assuming that $\overline{\mathbf{u}} = 0$ (with the implication $\overline{\mathbf{b}} = 0$),

$$\partial \mathbf{B}_0/\partial t = \nabla \wedge \overline{(\mathbf{u} \wedge \mathbf{b})} + \lambda \nabla^2 \mathbf{B}_0, \tag{2.8}$$

and this is the appropriate modification of (2.4) as an equation describing the evolution of the large-scale field $\mathbf{B}_0(\mathbf{x}, t)$.

The fluctuation field $\mathbf{b}(\mathbf{x}, t)$ may be calculated within the box V_a to the lowest order in the small parameter $\epsilon = l/L$ by neglecting variation of $\mathbf{B}_0$ throughout V_a, i.e. by treating $\mathbf{B}_0$ as locally uniform. The equation for $\mathbf{b}$ is then†

$$\partial \mathbf{b}/\partial t = \nabla \wedge [\mathbf{u} \wedge (\mathbf{B}_0 + \mathbf{b})] + \lambda \nabla^2 \mathbf{b}. \tag{2.9}$$

The term $\nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0)$ has the character of a forcing term in this equation and it is responsible for the generation of the fluctuations $\mathbf{b}$ on scales comparable with the scale l of $\mathbf{u}$.

Equation (2.9) is in general difficult to solve due to the presence of the random coefficient in one of the terms linear in $\mathbf{b}$, *viz.* $\nabla \wedge (\mathbf{u} \wedge \mathbf{b})$. However, under the condition $R_m \ll 1$,

$$|\mathbf{b}| = O(R_m)|\mathbf{B}_0| \tag{2.10}$$

(as may easily be verified *a posteriori*), so that $\nabla \wedge (\mathbf{u} \wedge \mathbf{b})$ may be neglected in (2.9) in comparison with $\nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0)$. We are then left with the more tractable equation,

$$\partial \mathbf{b}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0) + \lambda \nabla^2 \mathbf{b}. \tag{2.11}$$

The first objective is to solve equation (2.11) for $\mathbf{b}$ in terms of $\mathbf{u}$; and then to evaluate the term $\nabla \wedge \overline{(\mathbf{u} \wedge \mathbf{b})}$ in equation (2.8). It will be supposed that compressibility effects are unimportant, so that $\nabla \cdot \mathbf{u} = 0$, and, neglecting (for the moment) spatial variation of $\mathbf{B}_0$, (2.11) takes the form

$$\partial \mathbf{b}/\partial t = \mathbf{B}_0 \cdot \nabla \mathbf{u} + \lambda \nabla^2 \mathbf{b}. \tag{2.12}$$

If the forcing term $\mathbf{B}_0 \cdot \nabla \mathbf{u}$ were steady, then, after the disappearance of any irrelevant transients, the term $\partial \mathbf{b}/\partial t$ in (2.12) would be identically zero. In fact the term $\mathbf{B}_0 \cdot \nabla \mathbf{u}$ is unsteady, partly due to variation of $\mathbf{u}$ on a time-scale (the 'turnover' time)

$$t_0 = l/u_0, \tag{2.13}$$

and partly due to variation of $\mathbf{B}_0$ on some time-scale t_1 (as yet undetermined). It will be assumed at this stage that t_1 is not less that t_0 (actually, it will appear later, see (5.13), that $t_1 \gg t_0$), so that the effective time during which the term $\mathbf{B}_0 \cdot \nabla \mathbf{u}$

† Strictly, a term $-\nabla \wedge \overline{(\mathbf{u} \wedge \mathbf{b})}$ should be included, but this is obviously small compared with $\nabla \wedge (\mathbf{u} \wedge \mathbf{b})$ and may be omitted.

varies significantly is of order t_0. The term $\partial\mathbf{b}/\partial t$ arises only through the time-variation of the forcing term on this time-scale; hence

$$|\partial\mathbf{b}/\partial t| = O(bu_0/l), \tag{2.14}$$

whereas

$$|\lambda\nabla^2\mathbf{b}| = O(\lambda b/l^2). \tag{2.15}$$

Since the ratio is $O(R_m)$, it is consistent to neglect $\partial\mathbf{b}/\partial t$ in (2.12) giving the equation

$$\lambda\nabla^2\mathbf{b} = -\mathbf{B}_0\cdot\nabla\mathbf{u}, \tag{2.16}$$

wherein $\mathbf{B}_0$ is to be treated as uniform.

It is now convenient to use the Fourier–Stieltjes representation (Batchelor 1953, §2.5)

$$\mathbf{u}(\mathbf{x},t) = \int d\mathbf{Z}(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{b}(\mathbf{x},t) = \int d\mathbf{Y}(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.17}$$

in terms of which (2.16) becomes

$$d\mathbf{Y}(\mathbf{k},t) = \frac{i\mathbf{B}_0\cdot\mathbf{k}}{\lambda k^2}\,d\mathbf{Z}(\mathbf{k},t). \tag{2.18}$$

We are now in a position to calculate $\overline{\mathbf{u}\wedge\mathbf{b}}$. Evidently

$$\overline{\mathbf{u}\wedge\mathbf{b}} = \int \overline{d\mathbf{Z}^*(\mathbf{k},t)\wedge d\mathbf{Y}(\mathbf{k},t)}, \tag{2.19}$$

so that, using (2.18),

$$(\overline{\mathbf{u}\wedge\mathbf{b}})_i = \lambda^{-1}A_{ij}B_{0j}, \tag{2.20}$$

where

$$A_{ij} = i\int k^{-2}[\overline{d\mathbf{Z}^*(\mathbf{k},t)\wedge d\mathbf{Z}(\mathbf{k},t)}]_i k_j. \tag{2.21}$$

A_{ij} is a tensor determined by the statistics of the turbulence, and is uniform and steady only in so far as the turbulence is homogeneous and stationary.

Returning now to (2.8), and recognizing that $\mathbf{B}_0(\mathbf{x})$ *does* vary on scales of order L, we have

$$\frac{\partial B_{0i}}{\partial t} = \lambda^{-1}\epsilon_{ijk}\frac{\partial}{\partial x_j}(A_{kl}B_{0l}) + \lambda\nabla^2 B_{0i}, \tag{2.22}$$

and this is the equation that replaces (2.4) when the effects of the turbulence are taken into consideration. The tensor coefficient A_{kl} is placed under the operator $\partial/\partial x_j$ to allow for the possibility of inhomogeneity of the turbulence on scales much greater than a; in the derivation of (2.22) it was necessary only that the statistical properties of the turbulence should have negligible variation throughout the box V_a. The effects of the turbulence on the large-scale field are wholly summarized in the first term on the right-hand side of (2.22). The important difference between (2.22) and (2.2) is of course that the quantity A_{kl} is not a random quantity (like $\mathbf{u}(\mathbf{x},t)$) but a smoothed-out average property of the turbulence. In the particular case of homogeneous, stationary turbulence, on which attention will be focussed in the following sections, A_{kl} is independent of $\mathbf{x}$ and of t; and, dropping the suffix zero on the field $\mathbf{B}_0$, (2.22) becomes

$$\frac{\partial B_i}{\partial t} = \lambda^{-1}\epsilon_{ijk}A_{kl}\frac{\partial}{\partial x_j}B_l + \lambda\nabla^2 B_i. \tag{2.23}$$

3. Properties of the tensor A_{ij}

The Fourier–Stieltjes transform dZ_i is related to the spectrum tensor $\Phi_{ij}(\mathbf{k})$ of the turbulence by the equation

$$\Phi_{ij}(\mathbf{k}) = \lim_{d^3\mathbf{k}\to 0} \frac{\overline{dZ_i^*(\mathbf{k},t)\,dZ_j(\mathbf{k},t)}}{d^3\mathbf{k}}. \tag{3.1}$$

Hence (2.21) gives

$$A_{ij} = i\epsilon_{ikl}\int k^{-2}k_j\,\Phi_{kl}(\mathbf{k})d^3\mathbf{k}. \tag{3.2}$$

The spectrum tensor satisfies the condition of Hermitian symmetry,

$$\Phi_{kl}(\mathbf{k}) = \Phi_{lk}^*(\mathbf{k}), \tag{3.3}$$

so that

$$A_{ij}^* = -i\epsilon_{ikl}\int k^{-2}k_j\Phi_{lk}(\mathbf{k})\,d^3\mathbf{k} = -i\epsilon_{ilk}\int k^{-2}k_j\,\Phi_{kl}(\mathbf{k})\,d^3\mathbf{k} = A_{ij}, \tag{3.4}$$

i.e. A_{ij} is real (as is required by the relation (2.20)). Moreover $\Phi_{kl}(\mathbf{k})$ satisfies the incompressibility conditions

$$k_k\,\Phi_{kl}(\mathbf{k}) = 0, \quad k_l\,\Phi_{kl}(\mathbf{k}) = 0, \tag{3.5}$$

so that, for example,

$$\begin{aligned} A_{12}-A_{21} &= i\int k^{-2}k_2(\Phi_{23}-\Phi_{32})\,d^3\mathbf{k} - i\int k^{-2}k_1(\Phi_{31}-\Phi_{13})\,d^3\mathbf{k} \\ &= i\int [k^{-2}(k_2\Phi_{23}+k_1\Phi_{13}) - k^{-2}(k_2\,\Phi_{32}+k_1\,\Phi_{31})]\,d^3\mathbf{k} \\ &= i\int k^{-2}[-k_3\,\Phi_{33}+k_3\,\Phi_{33}]\,d^3\mathbf{k} = 0, \end{aligned}$$

and it follows that A_{ij} is symmetric. It is therefore possible to diagonalize A_{ij} by suitable choice of axes, so that say,

$$A_{ij} = \begin{pmatrix} \alpha & \cdot & \cdot \\ \cdot & \beta & \cdot \\ \cdot & \cdot & \gamma \end{pmatrix}, \tag{3.6}$$

where α, β, γ are real parameters determined by the statistical structure of the turbulence.

Suppose first that the turbulence is isotropic, i.e. its statistical properties are invariant with respect to rotation of the axes of reference and with respect to reflexions in the origin of reference. Then $\Phi_{ij}(\mathbf{k})$ takes the simple form,

$$\Phi_{ij}(\mathbf{k}) = \frac{E(k)}{4\pi k^4}(k^2\delta_{ij}-k_ik_j) = \Phi_{ij}^0(\mathbf{k}), \quad \text{say}, \tag{3.7}$$

and it is easy to see from (3.2) that in this case

$$A_{ij} = 0. \tag{3.8}$$

The conclusion is that to the lowest order in the small parameters ϵ and R_m, isotropic turbulence has no effect on the decay of a (large-scale) magnetic field distribution.

If we relax the condition of reflexional symmetry, then (Batchelor 1953, §3.3) we can have an additional term in the expression for the spectrum tensor, viz.

$$\Phi_{ij}(\mathbf{k}) = \Phi^0_{ij}(\mathbf{k}) + \frac{iF(k)}{8\pi k^4}\,\epsilon_{ijk}k_k, \tag{3.9}$$

where, by virtue of (3.3), $F(k)$ is real. The tensor (3.9) is still invariant with respect to rotation of the axes, but the second term changes sign on reflexion in the origin, i.e. on change from a right-handed to a left-handed co-ordinate system. Turbulence for which $F(k) \neq 0$ may be described as 'pseudo-isotropic'. Substitution of (3.9) in (3.2) gives

$$A_{ij} = \alpha\delta_{ij}, \tag{3.10}$$

where
$$\alpha = -\tfrac{1}{2}\int_0^\infty k^{-2}F(k)\,dk. \tag{3.11}$$

The lack of reflexional symmetry, represented by the function $F(k)$ is associated with the degree of right-handedness or left-handedness (or 'helicity') of the turbulence; the appropriate measure of this quantity for homogeneous turbulence (cf. Moffatt 1968) is

$$\overline{\mathbf{u}\,.\,\boldsymbol{\omega}} = i\int \overline{d\mathbf{Z}^*(\mathbf{k},t)\,.\,\mathbf{k}\wedge d\mathbf{Z}(\mathbf{k},t)} = -i\int k_i\,\epsilon_{ijk}\Phi_{jk}(\mathbf{k})\,d^3\mathbf{k}, \tag{3.12}$$

and, with Φ_{ij} given by (3.9),

$$\overline{\mathbf{u}\,.\,\boldsymbol{\omega}} = \frac{1}{8\pi}\int k^{-4}k_i\epsilon_{ijk}F(k)\,\epsilon_{jkl}k_l\,d^3\mathbf{k} = \int_0^\infty F(k)\,dk, \tag{3.13}$$

so that $F(k)dk$ may be regarded as the contribution to mean helicity of the turbulence from the element of wave-number space between the spheres of radii k and $k+dk$.

Suppose now that we weaken the symmetry conditions further, and suppose that the turbulence is axisymmetric (but without reflexional symmetry) about an axis in the direction of a unit vector $\boldsymbol{\lambda}$. The tensor $\Phi_{ij}(\mathbf{k})$ (and so the tensor A_{ij}) is then invariant with respect to rotations of the axes of reference about this axis. Hence $\boldsymbol{\lambda}$ must coincide with one of the principal axes of A_{ij}, say $\boldsymbol{\lambda} = (1,0,0)$. The general form for $\Phi_{ij}(\mathbf{k})$ satisfying the conditions (3.3) and (3.5) is

$$\begin{aligned}\Phi_{ij}(\mathbf{k}) = {}& Ak_ik_j + B\lambda_i\lambda_j + C\delta_{ij} + D(k_i\lambda_j + k_j\lambda_i) + iG\epsilon_{ijk}\lambda_k + iH\epsilon_{ijk}k_k \\ & + iM[(\mathbf{k}\wedge\boldsymbol{\lambda})_ik_j - (\mathbf{k}\wedge\boldsymbol{\lambda})_jk_i] + iN[(\mathbf{k}\wedge\boldsymbol{\lambda})_i\lambda_j - (\mathbf{k}\wedge\boldsymbol{\lambda})_j\lambda_i],\end{aligned} \tag{3.14}$$

where $A, B, \ldots, N$ are real functions of $\mathbf{k}\,.\,\mathbf{k}$ and $\mathbf{k}\,.\,\boldsymbol{\lambda}$, i.e. of k and k_1, related by

$$\left.\begin{aligned} k^2A + C + k_1D &= 0, \\ k_1B + k^2D &= 0, \\ -G + k^2M + k_1N &= 0. \end{aligned}\right\} \tag{3.15}$$

Substitution of (3.14) in (3.2) and simplification using (3.15) leads to

$$A_{ij} = 2\int\left[\frac{k_ik_j}{k^2}(H + k_1M + N) + \frac{\delta_{i1}\delta_{j1}}{k^2}G\right]d^3\mathbf{k}. \tag{3.16}$$

It is evident from this expression that (as we know already) $A_{22} = A_{33}$, but that A_{11} may be quite different in magnitude (and possibly in sign). Once again, the only contributions to A_{ij} come from terms of $\Phi_{ij}(\mathbf{k})$ which change sign on reflexion in the origin.

It is not necessary to go into similar details for the general case of homogeneous turbulence with no directional symmetry, for which the most general form of spectrum tensor consists of 31 terms of which 21 change sign on reflexion in the origin (Batchelor 1953, § 3.3). These 21 terms give (in general) unequal contributions to the principal values α, β, γ of A_{ij}. Comparison of (3.2) with the equation

$$\overline{u_i u_j} = \int \Phi_{ij}(\mathbf{k})\, d^3\mathbf{k} \tag{3.17}$$

suggests that A_{ij} is determined by the structure of the energy-containing eddies (with perhaps a little weighting at the low-k end of the spectrum), and so, on dimensional grounds,

$$(\alpha, \beta, \gamma) = (\alpha_0, \beta_0, \gamma_0)\, l u_0^2, \tag{3.18}$$

where α_0, β_0 and γ_0 are dimensionless constants (positive or negative) of order unity.

It is evident that the effect of turbulence on a large-scale magnetic field depends critically on whether the turbulence has reflexional symmetry in a point, and it may be as well to consider briefly whether a lack of reflexional symmetry is a likely state of affairs in geophysical and astrophysical turbulence. In general such turbulence arises as a result of an instability of a mean flow, the instability being frequently driven by buoyancy forces. It seems likely that this turbulence can lack reflexional symmetry only if the mean fields (velocity, temperature, etc.) themselves exhibit some lack of reflexional symmetry. The simplest, and most frequent, example arises in the case of turbulent thermal convection in a rotating fluid. The rotation vector and the direction of mean heat flux are together sufficient to give a definite right-handedness or left-handedness to the system, and it seems likely that a corresponding property will be represented in the statistics of the turbulence.

4. The helicity effect; physical interpretation

In the pseudo-isotropic case, $A_{kl} = \alpha\delta_{kl}$, and equation (2.22) becomes

$$\frac{\partial \mathbf{B}_0}{\partial t} = \frac{\alpha}{\lambda} \nabla \wedge \mathbf{B}_0 + \lambda \nabla^2 \mathbf{B}_0. \tag{4.1}$$

The term $(\alpha/\lambda)\nabla \wedge \mathbf{B}_0$ in (4.1) represents a tendency to generate magnetic field in the direction opposite (if $\alpha < 0$) to that of the large-scale current $\mathbf{J} = \mu\nabla \wedge \mathbf{B}_0$, and it is of interest to examine the physical mechanism underlying this effect. Suppose that we choose axes $Oxyz$ at a point in the fluid so that, locally,

$$\mathbf{J}_0 = (0, 0, J_0), \quad \mathbf{B}_0 = -\mu J_0(y, 0, 0), \tag{4.2}$$

and consider the action of a typical 'helicity wave'

$$\mathbf{u} = u_0(0, \sin(kx - \sigma t), \cos(kx - \sigma t)) = \mathrm{Re}\, \hat{\mathbf{u}}\, e^{i(kx - \sigma t)}, \tag{4.3}$$

where $\hat{\mathbf{u}} = u_0(0, -i, 1)$, and, to be definite, $k > 0$, $\sigma > 0$. For this motion,

$$\boldsymbol{\omega} = k\mathbf{u}, \quad \overline{\mathbf{u}.\boldsymbol{\omega}} = ku_0^2 > 0, \tag{4.4}$$

(the average being with respect to x or t). The streamlines (and vortex lines) of the motion are straight, but the particle paths (when $\sigma \neq 0$) are circular. If we imagine the vortex lines closed at large values of $|y|$ and $|z|$, then they are linked in the manner that gives positive helicity (Moffatt 1968).

According to (2.12), this motion generates a perturbation field

$$\mathbf{b} = \mathrm{Re}\, \hat{\mathbf{b}} e^{i(kx-\sigma t)}, \quad \hat{\mathbf{b}} = \frac{ikB_0}{\lambda k^2 - i\sigma} \hat{\mathbf{u}}. \tag{4.5}$$

If $\sigma \ll \lambda k^2$ (as is the case if $\sigma \approx t_0^{-1}$, $k \approx l^{-1}$, $R_m \ll 1$) then $\hat{\mathbf{b}} \approx (iB_0/\lambda k)\hat{\mathbf{u}}$, and so

$$\mathbf{b} = \frac{B_0 u_0}{\lambda k}(0, \cos(kx - \sigma t), -\sin(kx - \sigma t)). \tag{4.6}$$

The lines of force of the perturbed field $\mathbf{B}_0 + \mathbf{b}$ are *left-handed* helices, all with the same curvature (figure 2).† The total field in the region between the lines AC, A′C′ receives contributions from all the lines of force which penetrate that region; and it is evident that the unequal cancellation of contributions to the z-component of the generated field from neighbouring lines of force can lead to an average (negative) component of $\mathbf{B}$ in the z-direction.

This generation of B_z is assured, however, only if the direction of $\mathbf{u}$ (indicated by the small arrows) is such as to reinforce the convection of the field into the region between AC, A′C′. This is so in the case $\sigma \ll \lambda k^2$ considered here, for which

$$\left.\begin{aligned} \overline{\mathbf{u} \wedge \mathbf{b}} &= -(B_0 u_0^2/\lambda k)(0, 0, 1), \\ \nabla \wedge (\overline{\mathbf{u} \wedge \mathbf{b}}) &= (\alpha/\lambda)\nabla \wedge \mathbf{B}_0, \quad \alpha = -u_0^2/k. \end{aligned}\right\} \tag{4.7}$$

Consider, however, the other limit $\sigma \gg \lambda k^2$, for which $\hat{\mathbf{b}} \approx -(kB_0/\sigma)\hat{\mathbf{u}}$, and $\mathbf{b} = -(kB_0/\sigma)\mathbf{u}$, $\mathbf{u} \wedge \mathbf{b} = 0$. The lines of force of the perturbed field $\mathbf{B}_0 + \mathbf{b}$ are still helices, but the motion $\mathbf{u}$ does not reinforce the distortion, and so an average B_z does not materialize. It is evident that $\overline{\mathbf{u} \wedge \mathbf{b}}$ is non-zero only if $\mathbf{u}$ and $\mathbf{b}$ are out of phase, and this arises only if the process of generation of $\mathbf{b}$ has a dissipative character.

Likewise, a fully turbulent motion with non-zero helicity (say positive) will distort lines of force into 'random helices' with negative mean screw. In the case $R_m \ll 1$ treated in this paper, the expression for $\mathbf{u} \wedge \mathbf{b}$ contains contributions of the form (4.7) for each $\mathbf{k}$ represented in the Fourier decomposition of the $\mathbf{u}$-field, together with terms periodic in $\mathbf{x}$ which vanish when the average $\overline{\mathbf{u} \wedge \mathbf{b}}$ is taken. The cancellation effect between neighbouring lines of force will be incomplete when the initial field is non-uniform, and the generation of a mean component in the direction of $-\nabla \wedge \mathbf{B}_0$ will result. This effect is so intimately related to the

† Note that

$$\overline{(\mathbf{B}_0 + \mathbf{b}).\nabla \wedge (\mathbf{B}_0 + \mathbf{b})} = \overline{\mathbf{b}.(\nabla \wedge \mathbf{b})} = B_0^2 u_0^2/k\lambda^2 > 0$$

which might suggest (wrongly) a right-handed helical structure.

helicity of the background turbulence that the term 'helicity effect' would seem appropriate.†

The orders of magnitude of the two terms in (4.1) contributing to the rate of change of $\mathbf{B}$ are

$$\left|\frac{\alpha}{\lambda}\nabla\wedge\mathbf{B}\right| = O\left(\frac{\alpha B}{\lambda L}\right), \quad |\lambda\nabla^2\mathbf{B}| = O\left(\frac{\lambda B}{L^2}\right),$$

so that

$$\frac{|(\alpha/\lambda)\nabla\wedge\mathbf{B}|}{|\lambda\nabla^2\mathbf{B}|} = O\left(\frac{\alpha L}{\lambda^2}\right). \tag{4.8}$$

Hence, if $L \gg \lambda^2/\alpha$, the development of the field will be dominated by the turbulence term, at any rate as long as the scale of the field remains of order L, or greater.

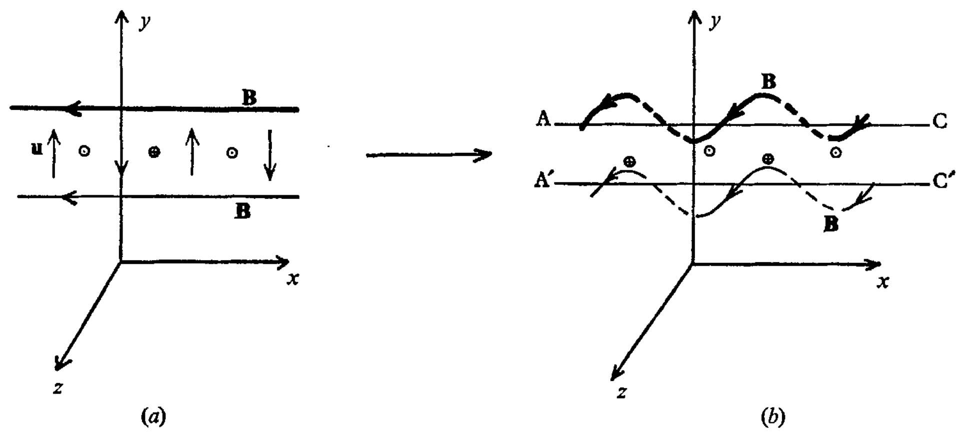

FIGURE 2. Distortion of initially straight lines of force by the helicity wave (4.3). The initial gradient of $\mathbf{B}$ is indicated by the different thickness of the lines of force. The velocity is indicated by the symbols $\uparrow\ \odot\ \downarrow\ \oplus$ in (*a*), where $\odot$ means into the paper and $\oplus$ means out of the paper. The lines of force are distorted into left-handed helices, and consideration of all the contributions to $\mathbf{B}$ from within the region in (*b*) between the lines AC, A′C′ indicates a net generation of B_z (in the negative z-direction) due to unequal cancellation of contributions from neighbouring lines of force.

The equation for magnetic energy density, from (4.1), is

$$\frac{\partial}{\partial t}\tfrac{1}{2}\mathbf{B}^2 = \frac{\alpha}{\lambda}\mathbf{B}.\nabla\wedge\mathbf{B} + \lambda\left[\frac{\partial}{\partial x_j}\left(B_i\frac{\partial}{\partial x_j}B_i\right) - \left(\frac{\partial B_i}{\partial x_j}\right)^2\right].$$

In the case of a random homogeneous field, this gives

$$\frac{d}{dt}\langle\tfrac{1}{2}\mathbf{B}^2\rangle = \frac{\alpha}{\lambda}\langle\mathbf{B}.(\nabla\wedge\mathbf{B})\rangle - \lambda\left\langle\left(\frac{\partial B_i}{\partial x_j}\right)^2\right\rangle, \tag{4.9}$$

(where the brackets $\langle\ldots\rangle$ signify a space average‡ over scales much greater than L), so that the magnetic energy density is affected by the turbulence only through the appearance of a mean helicity $\langle\mathbf{B}.(\nabla\wedge\mathbf{B})\rangle$ of the field $\mathbf{B}$. That such a mean helicity must appear is evident from the equation,

$$\frac{d}{dt}\langle\mathbf{B}.(\nabla\wedge\mathbf{B})\rangle = \frac{2\alpha}{\lambda}\langle(\nabla\wedge\mathbf{B})^2\rangle + \lambda\langle\mathbf{B}.\nabla^2(\nabla\wedge\mathbf{B}) + (\nabla\wedge\mathbf{B}).\nabla^2\mathbf{B}\rangle, \tag{4.10}$$

† The effect has been described as the 'α-effect' by Steenbeck *et al.* (1966).

‡ Or equivalently, an average over an ensemble of realizations of the $\mathbf{B}$-field.

derivable from (4.1) together with its curl. The first term on the right represents a production of mean helicity with the same sign as α, i.e. opposite to that of the mean helicity $\overline{\mathbf{u}.\boldsymbol{\omega}}$ of the background turbulence. The case of a localized 'blob' of magnetic field (on the scale L) is similar, the brackets $\langle \ldots \rangle$ in equations (4.9) and (4.10) being then replaced by $\int \ldots dv$, the integral being throughout the whole extent of the blob.

5. Dynamo action in the pseudo-isotropic case

(i) *Evolution of a single Fourier component of* $\mathbf{B}(\mathbf{x}, t)$

As a preliminary to the study of the evolution either of a homogeneous random $\mathbf{B}$-field or of a localized $\mathbf{B}$-field, it is appropriate to examine the evolution of a single Fourier component of the field; equation (4.1) evidently admits solution of the form

$$\hat{\mathbf{B}} e^{\omega t} e^{i\mathbf{K}.\mathbf{x}}, \quad \mathbf{K}.\hat{\mathbf{B}} = 0, \tag{5.1}$$

where

$$(\omega + \lambda K^2)\, \hat{\mathbf{B}} = \frac{i\alpha}{\lambda} \mathbf{K} \wedge \hat{\mathbf{B}}. \tag{5.2}$$

(The symbol $\mathbf{K}$ will be used for the wave-vector of any Fourier component of the $\mathbf{B}$-field, with the implicit understanding that $|\mathbf{K}| \ll |\mathbf{k}|$ where $\mathbf{k}$ is any wave-vector at which there is significant contribution to the background turbulence). With $\hat{\mathbf{B}} = \hat{\mathbf{B}}_r + i\hat{\mathbf{B}}_i$, the real and imaginary parts of (5.2) are

$$\left.\begin{aligned} (\omega + \lambda K^2)\, \hat{\mathbf{B}}_r &= -(\alpha/\lambda)\mathbf{K} \wedge \hat{\mathbf{B}}_i, \\ \text{and} \qquad (\omega + \lambda K^2)\, \hat{\mathbf{B}}_i &= (\alpha/\lambda)\mathbf{K} \wedge \hat{\mathbf{B}}_r, \end{aligned}\right\} \tag{5.3}$$

so that

$$(\omega + \lambda K^2)^2 \begin{pmatrix} \hat{\mathbf{B}}_r \\ \hat{\mathbf{B}}_i \end{pmatrix} = \left(\frac{\alpha}{\lambda}\right)^2 K^2 \begin{pmatrix} \hat{\mathbf{B}}_r \\ \hat{B}_i \end{pmatrix}, \tag{5.4}$$

and, for a non-trivial solution,

$$\omega = -\lambda K^2 \pm (\alpha K/\lambda) = \omega_1, \omega_2, \quad \text{say.} \tag{5.5}$$

From (5.3) we then have

$$\hat{\mathbf{B}}_i = \mp \hat{\mathbf{K}} \wedge \hat{\mathbf{B}}_r, \tag{5.6}$$

where $\hat{\mathbf{K}} = \mathbf{K}/K$, so that $|\hat{\mathbf{B}}_i| = |\hat{\mathbf{B}}_r|$ and $\hat{\mathbf{B}}_i . \hat{\mathbf{B}}_r = 0$, for each mode. The solution corresponding to the initial condition $\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0 e^{i\mathbf{K}.\mathbf{x}}$ is

$$\mathbf{B}(\mathbf{x}, t) = [\tfrac{1}{2}(\mathbf{B}_0 + i\hat{\mathbf{K}} \wedge \mathbf{B}_0)\, e^{\omega_1 t} + \tfrac{1}{2}(\mathbf{B}_0 - i\hat{\mathbf{K}} \wedge \mathbf{B}_0)\, e^{\omega_2 t}]\, e^{i\mathbf{K}.\mathbf{x}}. \tag{5.7}$$

It is evident from (5.5) that if

$$K < \alpha/\lambda^2 = K_c, \tag{5.8}$$

say, then the eigenvalue ω_1 is positive, and the corresponding solution (5.1) grows exponentially in time. The parabolas

$$\omega_1, \omega_2 = \lambda K(\pm K_c - K) \tag{5.9}$$

are shown in figure 3. The maximum rate of amplification occurs at $K = \frac{1}{2}K_c$, and at this value

$$\omega_1 = \omega_{\max} = \tfrac{1}{4}\lambda K_c^2 = \alpha^2/4\lambda^3. \tag{5.10}$$

Note that, with $\alpha = O(u_0^2 l)$, $K_c = O(u_0^2 l/\lambda^2)$, so that

$$K_c l = O(R_m^2). \tag{5.11}$$

Hence amplification will occur (in general) only on length scales of order $R_m^{-2} l$ and greater.

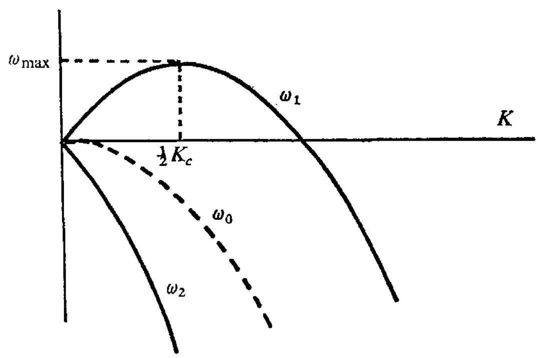

FIGURE 3. Variation of ω_1 and ω_2 as functions of K. The decay rate $\omega_0 = -\lambda K^2$ that would occur in the absence of turbulence is shown also for comparison.

The assumption following (2.13) concerning the time-scale of the large-scale field may now be vindicated. A minimum estimate for this time-scale (i.e. assuming maximum rate of change of $\mathbf{B}$) is

$$t_1 = O\left(\frac{1}{\omega_{\max}}\right) = O\left(\frac{\lambda^3}{u_0^4 l^2}\right). \tag{5.12}$$

Hence

$$t_1/t_0 = O(R_m^{-3}), \tag{5.13}$$

and this is certainly large (as assumed) when $R_m \ll 1$. Note that waves for which $K \gg K_c$ have $\omega_{1,2} \sim -\lambda K^2$, and are negligibly affected by the turbulence.

(ii) *Evolution of a random* $\mathbf{B}$*-field*

Suppose now that $\mathbf{B}(\mathbf{x}, 0)$ (and so $\mathbf{B}(\mathbf{x}, t)$) is itself a homogeneous random function of $\mathbf{x}$. Let

$$\mathbf{B}(\mathbf{x}, t) = \int d\mathbf{W}(\mathbf{K}, t)\, e^{i\mathbf{K}.\mathbf{x}}, \tag{5.14}$$

wherein $d\mathbf{W}(\mathbf{K}, t)$ evolves according to sub-section (i) above; then

$$B_i(\mathbf{x}, t) = \int Q_{ij}(\mathbf{K}, t)\, dW_j(\mathbf{K}, 0)\, e^{i\mathbf{K}.\mathbf{x}}, \tag{5.15}$$

where

$$Q_{ij}(\mathbf{K}, t) = \tfrac{1}{2}(e^{\omega_1 t} + e^{\omega_2 t})\,\delta_{ij} - \tfrac{1}{2} i \epsilon_{ijk} \hat{K}_k (e^{\omega_1 t} - e^{\omega_2 t}), \tag{5.16}$$

and $\omega_1(K)$, $\omega_2(K)$ are as given in (5.5) or (5.9). Note that $Q_{ij}(\mathbf{K}, t)$ satisfies the condition, necessary for the reality of $\mathbf{B}(\mathbf{x}, t)$,

$$Q_{ij}(-\mathbf{K}, t) = Q_{ij}^*(\mathbf{K}, t). \tag{5.17}$$

Clearly the development of $B_i(\mathbf{x}, t)$ is ultimately dominated by contributions to the integral (5.15) from the region $K < K_c$ of K-space for which $\omega_1 > 0$, and since $\omega_2 < 0$, we may take (in that region)

$$Q_{ij}(\mathbf{K}, t) \sim \tfrac{1}{2}(\delta_{ij} - i\epsilon_{ijk}\hat{K}_k)\, e^{\omega_1 t}. \tag{5.18}$$

The spectrum tensor of the field $\mathbf{B}(\mathbf{x}, t)$ is given by

$$\Gamma_{ij}(\mathbf{K}, t) = \lim_{d^3\mathbf{K}\to 0} \frac{\langle dW_i^*(\mathbf{K}, t)\, dW_j(\mathbf{K}, t)\rangle}{d^3\mathbf{K}} = Q_{ik}^*(\mathbf{K}, t)\, Q_{jl}(\mathbf{K}, t)\, \Gamma_{kl}(\mathbf{K}, 0). \quad (5.19)$$

It will be sufficient to consider the case in which $\mathbf{B}(\mathbf{x}, 0)$ is (statistically) pseudo-isotropic, so that

$$\Gamma_{kl}(\mathbf{K}, 0) = M(K)(\delta_{kl} - \hat{K}_k \hat{K}_l) - iN(K)\epsilon_{klm}\hat{K}_m. \quad (5.20)$$

Substitution of (5.16) and (5.20) in (5.19) leads, after some simplification, to

$$\begin{aligned}\Gamma_{ij}(\mathbf{K}, t) = {} & \tfrac{1}{2}[M(K)(e^{2\omega_1 t} + e^{2\omega_2 t}) + N(K)(e^{2\omega_1 t} - e^{2\omega_2 t})]\,(\delta_{ij} - \hat{K}_i\hat{K}_j) \\ & - \tfrac{1}{2}i[N(K)(e^{2\omega_1 t} + e^{2\omega_2 t}) + M(K)(e^{2\omega_1 t} - e^{2\omega_2 t})]\epsilon_{ijq}\hat{K}_q, \end{aligned} \quad (5.21)$$

and, for $K < K_c$ and $\omega_{\max} t \gg 1$, this gives

$$\Gamma_{ij}(\mathbf{K}, t) \sim \tfrac{1}{2}[M(K) + N(K)]\,(\delta_{ij} - \hat{K}_i\hat{K}_j - i\epsilon_{ijk}\hat{K}_k)e^{2\omega_1 t}. \quad (5.22)$$

The magnetic energy density is given by

$$\tfrac{1}{2}\langle \mathbf{B}^2\rangle = \tfrac{1}{2}\int \Gamma_{ii}(\mathbf{K}, t)\, d^3\mathbf{K}, \quad (5.23)$$

so that, ultimately, restricting the integration to the region $K < K_c$ which provides the growing contribution,

$$\tfrac{1}{2}\langle \mathbf{B}^2\rangle \sim \tfrac{1}{2}\int_0^{K_c} [M(K) + N(K)]\, e^{2\omega_1(K)t}\, 4\pi K^2 dK. \quad (5.24)$$

It may reasonably be assumed that $M(K)$ and $N(K)$ vary continuously in the range $(0, K_c)$. The function $4\pi K^2 M(K)$ is the initial magnetic energy spectrum function, and it may be expected to have a maximum at some wave-number of order L^{-1}, since the initial field has typical length scale L. The function $N(K)$ will be identically zero if the initial field is strictly isotropic; if not, it may be expected to have the same qualitative behaviour as $M(K)$. With $\omega_1(K) = \lambda K(K_c - K)$, it is evident from (5.24) that there is a preferential amplification of different contributions in the range $(0, K_c)$ and that ultimately the integral is dominated by contributions from the neighbourhood of $K = \frac{1}{2}K_c$. Asymptotic evaluation of the integral gives, for $\omega_{\max} t \gg 1$,

$$\begin{aligned}\tfrac{1}{2}\langle \mathbf{B}^2\rangle &\sim 2\pi(\tfrac{1}{2}K_c)^2\,[M(\tfrac{1}{2}K_c) + N(\tfrac{1}{2}K_c)]\, e^{2\omega_{\max} t}\int_{-\infty}^{\infty} e^{-2\lambda(K - \frac{1}{2}K_c)^2 t}\, dK \\ &= \tfrac{1}{2}\pi K_c^2\,[M(\tfrac{1}{2}K_c) + N(\tfrac{1}{2}K_c)]\,(\pi/2\lambda t)^{\frac{1}{2}}\, e^{2\omega_{\max} t}. \end{aligned} \quad (5.25)$$

To be quite specific, suppose that

$$N(K) = 0, \quad 4\pi K^2 M(K) = CK^4 e^{-L^2K^2}, \quad (5.26)$$

corresponding to an initially isotropic random $\mathbf{B}$-field with a simple spectrum function having a single maximum at a wave-number of order L^{-1}. Then

$$\tfrac{1}{2}\langle \mathbf{B}^2\rangle \sim \tfrac{1}{4}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\, CK_c^4 \frac{1}{4\pi}\exp\left(-\tfrac{1}{4}L^2K_c^2\right)\left(\frac{1}{\lambda t}\right)^{\frac{1}{2}} \exp\left(\alpha^2 t/2\lambda^3\right). \quad (5.27)$$

With $\alpha = O(u_0^2 l)$, the doubling time for the magnetic energy is ultimately of order $R_m^{-3} t_0$ where $t_0 = l/u_0$ and the characteristic length scale of the field that ultimately dominates is of order K_c^{-1}, i.e. of order $R_m^{-2} l$. (This may be large or small compared with its initial scale $L = O(\epsilon^{-1} l)$.)

(iii) *Evolution of a localized* **B**-*field*

Suppose now that the initial field $\mathbf{B}(\mathbf{x}, 0)$ is that due to a localized current distribution $\mathbf{J}(\mathbf{x}, 0)$, i.e.

$$\nabla \wedge \mathbf{B}(\mathbf{x}, 0) = \mu \mathbf{J}(\mathbf{x}, 0), \quad \nabla . \mathbf{B}(\mathbf{x}, 0) = 0. \tag{5.28}$$

Such a field is in general $O(|\mathbf{x}|^{-3})$ as $|\mathbf{x}| \to \infty$ and its Fourier transform has an associated directional singularity at $\mathbf{K} = 0$. Writing

$$\mathbf{B}(\mathbf{x}, 0) = \int \hat{\mathbf{B}}(\mathbf{K}, 0)\, e^{i\mathbf{K}.\mathbf{x}} d^3\mathbf{K}, \tag{5.29}$$

it is known (by analogy with the problem studied by Phillips 1956) that, for $\mathbf{K} \to 0$,

$$\hat{B}_i(\mathbf{K}, 0) = (\delta_{ij} - \hat{K}_i \hat{K}_j) P_j + O(K), \tag{5.30}$$

where P_i is independent of **K**.

The Fourier transform again evolves as in (i) above, so that

$$B_i(\mathbf{x}, t) = \int Q_{ij}(\mathbf{K}, t)\, \hat{B}_j(\mathbf{K}, 0)\, e^{i\mathbf{K}.\mathbf{x}} d^3\mathbf{K}, \tag{5.31}$$

and the total energy of the blob is

$$M = \frac{1}{2}\int \mathbf{B}^2 dV = \frac{1}{2}\int Q_{ij}(\mathbf{K}, t)\, Q^*_{il}(\mathbf{K}, t)\, \hat{B}_j(\mathbf{K}, 0)\, \hat{B}^*_l(\mathbf{K}, 0)\, d^3\mathbf{K}, \tag{5.32}$$

so that, ultimately,

$$M \sim \tfrac{1}{8} e^{2\omega_{\max} t} \int (\delta_{ij} - i\epsilon_{ijk}\hat{K}_k)(\delta_{il} + i\epsilon_{ilm}\hat{K}_m)\, \hat{B}_j(\tfrac{1}{2}\mathbf{K}_c, 0)$$
$$\times \hat{B}^*_l(\tfrac{1}{2}\mathbf{K}_c, 0)\, dA(\tfrac{1}{2}\mathbf{K}_c) . \int \exp\{-2\lambda(K - \tfrac{1}{2}K_c)^2\} dK,$$

where $\mathbf{K}_c = K_c \hat{\mathbf{K}}$, and $dA(\frac{1}{2}\mathbf{K}_c)$ indicates integration over the surface of the sphere $|\mathbf{K}| = \frac{1}{2}K_c$ in **K**-space. This simplifies to

$$M \sim \frac{1}{4}\left(\frac{\pi}{2\lambda t}\right)^{\frac{1}{2}} e^{2\omega_{\max} t} \int [|\hat{\mathbf{B}}_c|^2 - 2\hat{\mathbf{K}} . (\hat{\mathbf{B}}_{cr} \wedge \hat{\mathbf{B}}_{ci})]\, dA(\tfrac{1}{2}\mathbf{K}_c), \tag{5.33}$$

where
$$\hat{\mathbf{B}}_c (= \hat{\mathbf{B}}_{cr} + i\hat{\mathbf{B}}_{ci}) = \hat{\mathbf{B}}\,(\tfrac{1}{2}\mathbf{K}_c, 0).$$

6. Dynamo action for general homogeneous turbulence

When the principal values α, β, γ of A_{ij} are all different, (2.23) still admits plane wave solutions of the form (5.1); substitution gives

$$(\omega + \lambda K^2)\hat{B}_1 = (i/\lambda)(\gamma K_2 \hat{B}_3 - \beta K_3 \hat{B}_2), \tag{6.1}$$

and two other components given by cyclic permutation. For a non-trivial solution,

$$\begin{vmatrix} \omega+\lambda K^2 & \dfrac{i\beta}{\lambda}K^3 & -\dfrac{i\gamma}{\lambda}K_2 \\ -\dfrac{i\alpha}{\lambda}K_3 & \omega+\lambda K^2 & \dfrac{i\gamma}{\lambda}K_1 \\ \dfrac{i\alpha}{\lambda}K_2 & -\dfrac{i\beta}{\lambda}K_1 & \omega+\lambda K^2 \end{vmatrix} = 0, \tag{6.2}$$

a cubic with roots
$$\omega = -\lambda K^2, \tag{6.3}$$
and
$$\omega_1, \omega_2 = -\lambda K^2 \pm (1/\lambda)(\beta\gamma K_1^2 + \gamma\alpha K_2^2 + \alpha\beta K_3^2)^{\frac{1}{2}}. \tag{6.4}$$

The two roots given by (6.4) reduce to those given previously in (5.9) in the isotropic case $\alpha = \beta = \gamma$.

The root (6.3) is relevant if the right-hand side of (6.1) (and the two similar equations) vanishes, i.e. if
$$\frac{\hat{B}_1\alpha}{K_1} = \frac{\hat{B}_2\beta}{K_2} = \frac{\hat{B}_3\gamma}{K_3}. \tag{6.5}$$

Since the field must satisfy $\hat{\mathbf{K}}.\hat{\mathbf{B}} = 0$, this can arise only if
$$\frac{K_1^2}{\alpha} + \frac{K_2^2}{\beta} + \frac{K_3^2}{\gamma} = 0, \tag{6.6}$$

and this can happen only if α, β and γ are not all of the same sign. In this case, (6.6) defines a cone in **K**-space, and if **K** lies along a generator of this cone, then there is no interaction with the turbulence, and the wave decays as $e^{-\lambda K^2 t}$; this circumstance must clearly be regarded as exceptional.

If the roots (6.4) are substituted back in the three equations for the components of $\hat{\mathbf{B}}$, we obtain
$$\frac{\hat{B}_1}{\gamma K_2^2 + \beta K_3^2} = \frac{\hat{B}_2}{-\gamma K_1 K_2 \pm i K_3 Q} = \frac{\hat{B}_3}{-\beta K_1 K_3 \mp i K_2 Q}, \tag{6.7}$$
where
$$Q = +(\beta\gamma K_1^2 + \gamma\alpha K_2^2 + \alpha\beta K_3^2)^{\frac{1}{2}}. \tag{6.8}$$

For these modes, $\mathbf{K}.\hat{\mathbf{B}} = 0$ without restriction on the direction of **K**. If **K** does not satisfy (6.6), then the general behaviour may be represented by a sum of modes, *viz.*
$$\hat{\mathbf{B}}(t) = \mathbf{B}^{(1)}e^{\omega_1 t} + \mathbf{B}^{(2)}e^{\omega_2 t}, \tag{6.9}$$
the values of $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ being obtainable in terms of $\hat{\mathbf{B}}(0)$, using (6.7).

The roots (6.4) are complex (with negative real part) if $Q^2 < 0$, and they are real if $Q^2 > 0$. If, moreover,
$$Q^2 = \beta\gamma K_1^2 + \gamma\alpha K_2^2 + \alpha\beta K_3^2 > \lambda^4 K^4, \tag{6.10}$$
then the root ω_1 is real and positive, and the associated wave is amplified exponentially in time. If α, β and γ are all of the same sign, then this behaviour is qualitatively the same as in the isotropic case; the surface $Q^2 = \lambda^4 K^4$ is a closed ovoid intersecting the K_1, K_2 and K_3 axes in the points
$$\pm\frac{(\beta\gamma)^{\frac{1}{2}}}{\lambda^2}, \quad \pm\frac{(\gamma\alpha)^{\frac{1}{2}}}{\lambda^2}, \quad \pm\frac{(\alpha\beta)^{\frac{1}{2}}}{\lambda^2},$$
respectively, and a mode of wave-number **K** decays or amplifies according as the end-point of the vector **K** lies outside or inside this surface (figure 4).

If α, β and γ have different signs, suppose that $\alpha \geqslant \beta > 0 > \gamma$ (the case $\beta < 0$ being similar). The surface $Q^2 = \lambda^4 K^4$ intersects any plane through the K_3-axis in a figure-of-eight, and once again the mode decays or amplifies according as $\mathbf{K}$ lies outside or inside the surface. If $|\gamma| \gg \alpha$, β, then the surface is closely wrapped round the K_3-axis. In this case, only disturbances for which $\mathbf{K}$ is nearly aligned with the K_3-axis (and satisfying $K < (\alpha\beta)^{\frac{1}{2}}/\lambda^2$) will be amplified.

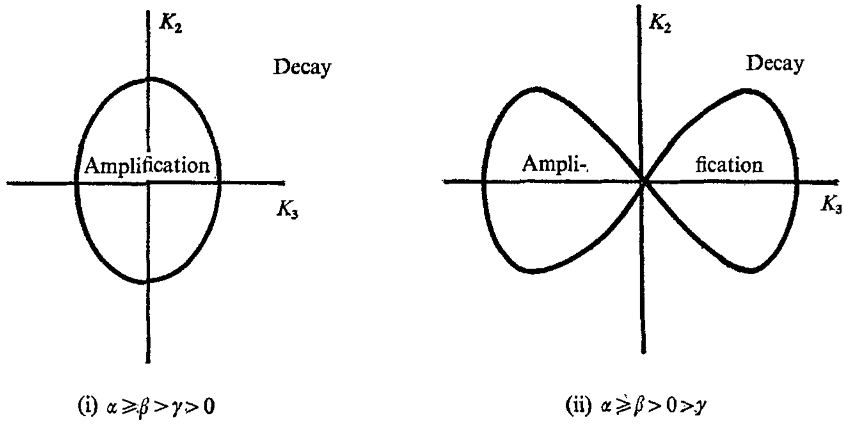

FIGURE 4. Intersection of the surface $Q^2 = \lambda^4 K^4$ with the $K_2 - K_3$ plane. In the axisymmetric case ($\alpha = \beta$) the surface is the surface of revolution obtained by rotating the curve about the K_3-axis. The region inside the surface is the region of $\mathbf{K}$-space in which Fourier components of the $\mathbf{B}$-field are amplified.

If $\mathbf{B}(\mathbf{x}, 0)$ is a random homogeneous function of $\mathbf{x}$, it is clear that, since the growth rates of its constituent Fourier components depend on the direction as well as the magnitude of their wave-vectors, anisotropy will develop even if the initial $\mathbf{B}$-field is isotropic. The anisotropy will be particularly strong if α, β and γ are very different in magnitude. For example, in the case $\gamma < 0$, $|\gamma| \gg \alpha, \beta$, (case (ii) in figure 4) the decay of all Fourier components for which

$$K_1, K_2 \geqslant (\alpha/|\gamma|)K_3, (\beta/|\gamma|)K_3$$

implies a systematic increase in the characteristic length scale of the field in the x and y directions. Ultimately, the field will vary significantly only in the z-direction.

Similar considerations apply to the case of a localized initial field. In this case if $\gamma < 0$, and $|\gamma| \gg \alpha, \beta$, then the lateral extent of the blob in the x and y directions will increase as $(\lambda t)^{\frac{1}{2}}$, since the dominant process affecting Fourier components with wave-vectors in the corresponding directions in $\mathbf{K}$-space is the conventional ohmic decay. The length scale in the z-direction on the other hand will settle down to $O(\lambda^2/|\gamma|)$ as in §5.

7. Discussion

It has been established in the foregoing sections that turbulence having a dominant scale l is capable of systematically amplifying a magnetic field whose initial length scale L is large compared with l; it was assumed that $R_m \ll 1$; and a sufficient condition for dynamo action is then that at least two of the principal values α, β, γ of the tensor A_{ij}, defined by (3.2) in terms of the spectrum tensor of the turbulence, should be non-zero, a condition that is generally satisfied in turbulence that lacks reflexional symmetry.

The assumption $L/l \gg 1$ is to some extent merely a convenience in describing the problem at the outset. If the initial field $\mathbf{B}(\mathbf{x}, 0)$ is a homogeneous random function of $\mathbf{x}$ and if the condition $L/l \gg 1$ is not satisfied, then any of its Fourier components for which $|\mathbf{K}| = O(l^{-1})$ will decay quite rapidly (since $R_m \ll 1$) leaving only those components (possibly extremely weak) for which $|\mathbf{K}| \ll l^{-1}$. These would decay slowly in the absence of turbulence; but the effect of the turbulence is to regenerate them through a coherent interaction between the velocity and the small-scale magnetic fluctuations.

The characteristic scale of the amplifying magnetic field ultimately settles down to $O(R_m^{-2}l)$. There is an obvious difficulty in designing a laboratory experiment to test the theory; with $R_m \approx 10^{-1}$ (in mercury, this would require a turbulent Reynolds number $R \approx 10^5$) and with $l \approx 10^{-2}$ m, a magnetic field would be amplified only on scales of order 1 m and greater. The possibility of an experiment on mercury (or any other liquid metal) on this scale is remote.

In situations of interest in astrophysics, the magnetic Reynolds number may often be of order unity, or much greater. Under these circumstances, it is possible that the low wave-number Fourier components of the $\mathbf{B}$-field (i.e. those for which $|\mathbf{K}|l \ll 1$) still grow exponentially in time (and Roberts 1969 has provided evidence in the case of velocity fields that are periodic in space and in time that this is the case), but the evolution of the field is likely to be dominated by the growth of Fourier components for which $|\mathbf{K}|l = O(1)$ and greater, and the approach described in this paper is not adequate to treat such behaviour.

The crucial stage of this work is contained in § 2; because, once (2.23) is accepted, dynamo action follows as an inevitable consequence without further approximation. The analysis of § 2 has been presented in a manner that makes some appeal to physical intuition, but minimal appeal to mathematical formalism. The analysis may be formalized by introducing a space variable $\mathbf{X} = R_m^2\mathbf{x}$, and a time variable $T = R_m^3 t$ (these being suggested by the results (5.11) and (5.13)), and by then allowing $\mathbf{B}$ to depend (independently) on $\mathbf{x}$, $\mathbf{X}$, t and T. The dependence on $\mathbf{x}$, t describes the fluctuations on scales characteristic of the turbulence, and the dependence on $\mathbf{X}, T$ describes the long-time development of the large-scale structure of the field. To lowest order in R_m, this approach leads to precisely the equations (2.8) and (2.16) given in § 2; and it can also yield systematic higher approximations. If the helicity spectrum $F(k)$ is identically zero, then the first effect of the turbulence at low R_m must be sought at these higher levels and at this stage, higher order spectral tensors (than the second) of the turbulence enter the analysis. Developments in this direction perhaps merit further study.

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